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DERIVATION OF CUMULATIVE
MULTI-PING DETECTION PROBABILITY MODEL

By

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This technical note describes the development of a computer program for the calculation of multi-ping probability of detection as a part of a total computer model for sonar prediction. This memorandum has been prepared because it is believed that the information may be useful in this form to others at Naval Undersea Research and Development Center. It should not be construed as a report as its only function is to present a portion of the work as information for others. This technical note supersedes NUWC TN 90.

INTRODUCTION

One type of detection criterion of interest for use in sonar systems is a "counting criterion" where multiple looks at a target area are available. The receiver output for each look or ping is compared with a threshold and generates a binary one output if that threshold is exceeded or a binary zero if it fails to exceed the threshold. The multiple looks will now appear as a binary sequence and the "counting criterion" examines a group of length m, counting the number of ones in that group. If the number of ones, or threshold crossings, equals or exceeds the integer n the detection criterion is satisfied and a target is declared present. Otherwise another ping in the sequence is examined. This counting criterion is often called an n out of m criterion. Since successive groups of the m pings have $m-1$ pings in common the probability of satisfying the n of m criterion at any given point of the sequence is dependent upon the $m-1$ past outcomes. Such a sequence is often called a Markov chain of $m-1$ order. It is of interest to examine the joint probability of such overlapping groups of independent pings and to determine the probability that n or more of the group (in any order) exceed a fixed threshold, thus qualifying that group as satisfying an n out of m detection criterion. In addition, the cumulative probability that any group in the sequence satisfies the n of m criterion is desired. Since adjacent groups have many pings in common the groups themselves are not independent, even though the pings are assumed to be. It is assumed that the single ping probability of exceeding the threshold, P_k, is available and all solutions will be in terms of the single ping detection probability.
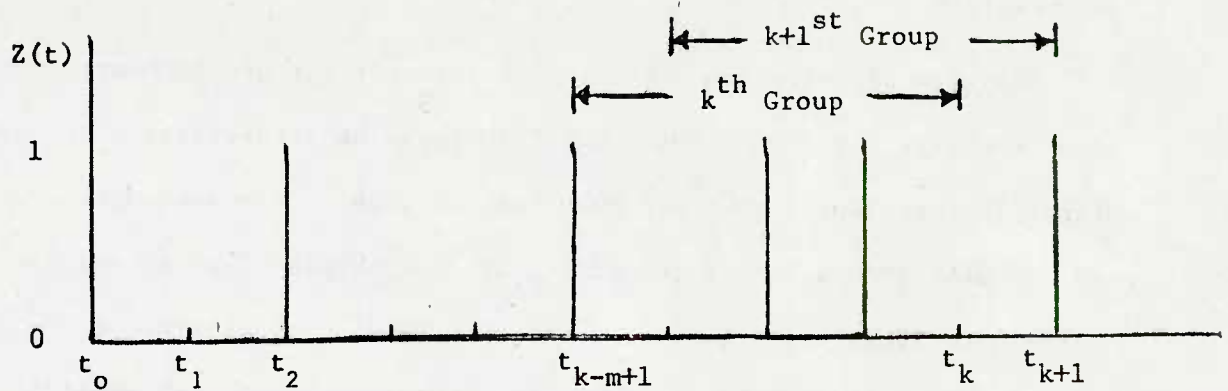
Figure 1: Binary Output of Threshold Receiver

JOINT PROBABILITY FOR K^{th} GROUP

Z(t) is a binary function which represents the sequence of single ping outputs. If the kth ping which is received at time t_k exceeds a fixed receiver threshold, $Z(t_k)$ is equal to one; otherwise, $Z(t_k)$ is zero. P_k is the single ping probability that the kth pulse exceeds the threshold and thus the probability that $Z(t_k)$ will equal one. We now examine a group of m adjacent pulses consisting of the $k-m+1^{st}$ to the kth and write down the probability that $Z(t_k)$ for this group will contain exactly n ones and $m-n$ zeros under the assumption of independence between pings.

If the P_k's were equal, the expression for the probability of exactly n out of m is the familiar binomial form:

$$P\{n/m,\ k\} = \binom{m}{n} P^n (1-P)^{m-n}, \quad P = P_k \quad \text{for all } k$$

and the probability of n or more out of m is

$$P\{\geq n/m,\ k\} = \sum_{j=n}^{m} \binom{m}{j} P^j (1-P)^{m-j}$$

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For the more general case of unequal P_k's, $P\{n/m,k\}$ is the sum of all possible products of n different P_k's and $m-n$ different $(1-P_k)$'s. Thus, if $n = 3$, $m = 4$.

$$P\{3/4,k\} = P_k\ P_{k-1}\ P_{k-2}\ (1-P_{k-3}) + P_k\ P_{k-1}\ (1-P_{k-2})\ P_{k-3}$$

$$+ P_k\ (1-P_{k-1})\ P_{k-2}\ P_{k-3} + (1-P_k)\ P_{k-1}\ P_{k-2}\ P_{k-3}.$$

The number of terms in the sum is

$$\binom{m}{n} \triangleq \frac{m!}{n!\ (m-n)!}$$

In the above example $P\{\geq 3/4,k\}$ is given by

$$P\{\geq 3/4,k\} = P\{3/4,k\} + P\{4/4,k\}$$

$$= P_k\ P_{k-1}\ P_{k-2}\ (1-P_{k-3}) + P_k\ P_{k-1}\ (1-P_{k-2})\ P_{k-3}$$

$$+ P_k\ (1-P_{k-1})\ P_{k-2}\ P_{k-3} + (1-P_k)\ P_{k-1}\ P_{k-2}\ P_{k-3}$$

$$+ P_k\ P_{k-1}\ P_{k-2}\ P_{k-3}$$

If $P_k = P$ the above expression reduces to

$$P\{\geq 3/4, k\} = P^3 (1-P) + P^3 (1-P) + P^3 (1-P) + P^3 (1-P) + P^4$$

$$= 4 P^3 (1-P) + P^4$$

$$= \binom{4}{3} P^3 (1-P) + \binom{4}{4} P^4 = \sum_{k=3}^{4} \binom{4}{k} P^k (1-P)^{4-k}$$

the same as previously noted.

Cumulative Probability:

It is of interest to calculate the probability that a string of pings of length k contains a sub-string of length m which satisfies the n of m criteria. This sub-string can be located anywhere in the string. This probability will be denoted by P_{cum} and is a monotonically increasing function of the string length k. To obtain a general algorithm for the calculation of P_{cum}, one approach is to generate all possible binary functions, $Z(t)$ of length k, selecting those that meet the criterion and calculate their probability and sum. Since the number of possible sequences is 2^k this procedure is not practical for large k. A second approach in which $P_{cum}(k)$ is a function of $P_{cum}(k-1)$ and a smaller set of binary functions will be described.

First a few definitions. Let $E(k)$ be the event that a given binary function of length k has n __or more__ ones (successes) occurring within the last m positions and $\overline{E(k)}$ the complement of that event. Thus, if $Z(t)$ is written as a binary number with the right most digit representing $Z(t_k)$ then for $k = 7$, $n = 2$, $m = 3$, the string 0010010 fails while 0010101 is in event $E(7)$. With this notation $P_{cum}(k)$ is the probability that

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E(k) occurs or E(k-1) or E(k-2) ... or E(m). Using DeMorgan's law

$$P_{cum}(k) = P(E(k) U E(k-1) U E(k-2) U \cdots U E(m)) \qquad *$$

$$= P(E(k-1) U E(k-2) U \cdots U E(m))$$

$$+ P(E(k) \cap \overline{E(k-1)} \cap \overline{E(k-2)} \cap \cdots \overline{E(m)})$$

$$P_{cum}(k) = P_{cum}(k-1) + P(E(k), \overline{E(k-1)}, \overline{E(k-2)}, \cdots, \overline{E(m)}).$$

Using the notation $P(A/B)$ for the probability of A given that B occurred, $P_{cum}(k)$ is given by

$$P_{cum}(k) = P_{cum}(k-1) + P(E(k) | \overline{E(k-1)}, \overline{E(k-2)}, \ldots \overline{E(m)})$$

$$\cdot P(\overline{E(k-1)}, \overline{E(k-2)}, \ldots \overline{E(m)})$$

but

$$P(\overline{E(k-1)}, \overline{E(k-2)}, \ldots \overline{E(m)}) = P(\overline{E(k-1) U E(k-2) U \ldots U E(m)})$$

$$= 1 - P_{cum}(k-1) \qquad \text{giving}$$

$$P_{cum}(k) = P_{cum}(k-1) + P(E(k) | \overline{E(k-1)}, \overline{E(k-2)}, \ldots, \overline{E(m)}) \cdot (1 - P_{cum}(k-1)).$$

---

*U is union, ∩ is intersection hereafter written with a comma

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The E(k) and E(k-m), have no common pings which determine the events success or failure; therefore,

$$P(E(k)|\overline{E(k-1)}, \overline{E(k-2)}, \ldots, \overline{E(m)}) = P(E(k)|\overline{E(k-1)}, \ldots, \overline{E(k-m+1)}).$$

Since the event E(k-m+1) depends on pings number (k-m+1) back to (k-m+1) - m+1 or (k-2m+2) and the event E(k) depends only on pings number k back to k-m+1 the $P(E(k)|\overline{E(k-1)}, \ldots, \overline{E(k-1-m+1)})$ may be determined by examining all possible binary sequences of length k - (k-2m+2) + 1 or 2m-1 ending with the kth. There are $2^{(2m-1)}$ possible sequences which can be partitioned into the following groups using n=2, m=3 as an example.

k-4	k-3	k-2	k-1	k	
1	1	0	1		**Group III:**
0	1	0	1		
1	1	1	0		All strings of length $(2m-2)$ such that at least one substring of length m or less contains n 1's.
0	1	1	0		
1	0	1	0		
0	0	1	1		
1	0	1	1		$n = 2$
0	1	1	1		$m = 3$
0	0	1	1		
1	1	0	0		
1	0	0	0		**Group II:**
0	1	0	0		All strings such that <u>no</u> substring of length m or less contains n 1's <u>and</u> there are less than $(n-1)$ 1's in the most recent (right most) substring of length $(m-1)$.
0	0	0	0		
0	0	1	0		**Group I:**
1	0	0	1		All strings such that no substring of length m or less contains n 1's <u>and</u> there are exactly $(n-1)$ 1's in the most recent substring of length $(m-1)$.
0	0	0	1		

$2^{(2m-2)}$ rows over span $2m-2$ columns.

Only those strings (of length $2m-2$) in Group I can result in the event $E(k)\,\vert\,\overline{E(k-1)}$ ... $\overline{E(k-m+1)}$ no matter what the outcome of the kth ping. If the kth ping is a success (1) the event is satisfied by the strings in Group I and <u>none</u> of the strings in Group II or Group III. If the outcome of the kth ping is a failure

(0) the event fails for all groups. Therefore, <u>all</u> successful strings of length 2m-1 are obtained by taking those in Group I and adding a one in the kth position in the example:

The probability of the event $E(k)|\overline{E(k-1)}, .. \overline{E(k-m+1)}$ may now be written in terms of the probability of the strings in Group I and Group II.

$$P(E(k)|\overline{E(k-1)}, .. , \overline{E(k-m+1)}) = \frac{P(E(k), E(\overline{k-1}), .., E(\overline{k-m+1}))}{P(\overline{E(k-1)}, ..., E(\overline{k-m+1}))}$$

$$= \frac{P(k) \cdot (\text{Sum of the probability of all strings in Group I})^*}{(\text{Sum of the probability of all strings in Group I or Group II})}$$

$$= \frac{P(k) \cdot P(Gp\ I)}{P(Gp\ I) + P(Gp\ II)}$$

*Note: In all successful strings the kth ping must be a 1 (success) and the k-m ping a 0 (failure).

In the example of n=2, m=3, k=5, and $P(i)$ the probability that the ith ping exceeds a fixed threshold.

$$P_k(GP\ I) = (1-P(1)) \cdot (1-P(2)) \cdot P(3) \cdot (1-P(4))$$

$$+ P(1) \cdot (1-P(2)) \cdot (1-P(3)) \cdot P(4)$$

$$+ (1-P(1)) \cdot (1-P(2)) \cdot (1-P(3)) \cdot P(4)$$

$$P_k(GP\ II) = P(1) \cdot (1-P(2)) \cdot (1-P(3)) \cdot (1-P(4))$$

$$+ (1-P(1)) \cdot P(2) \cdot (1-P(3)) \cdot (1-P(4))$$

$$+ (1-P(1)) \cdot (1-P(2)) \cdot (1-P(3)) \cdot (1-P(4))$$

and

$$P_{cum}(k) = P_{cum}(k-1) + \frac{P(k) \cdot P_k(GP\ I)}{P_k(GP\ I) + P_k(GP\ II)} \{1 - P_{cum}(k-1)\}$$

After the strings of length 2m-2 in Group I and Group II have been generated (for a fixed n and m) they are retained for calculating $P(E(k))|E(\overline{k-1}), .., E(\overline{k-m+1}))$ for all k > 2 m-1 by changing only the individual ping probabilities, $P(k)$, which enter into the calculation.

A general flow diagram for the calculation of $P_{cum}(k)$ for a given n and m is given in the appendix. The writing and check-out of this program was completed by Mr. K. A. Faucher of Code 556.

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Appendix: Flow Diagram

MAIN PROGRAM

P$_{cum}$ is an array of length L
containing the cumulative
probability.

P is an array of length L
containing the ping
probabilities.

S is an array of length Q
containing the sequences
in Group I.

T is an array of length V
containing the sequences
in Group II.

START

Input: N, M, L
P(k) k=1,2, ..L
K=0, Q=0

K = K + 1

K ≥ 2M - 1   No → Call Initialize

Yes

Call PGEN to
calculate the
prob {Gp I}

Call PGEN to
calculate the
prob {Gp II}

$P_{cum}(k) = P_{cum}(k-1)$
$+(1-P_{cum}(k-1))$
$*P(GP\ I)/P$
$(GP\ I) + P\ (GP\ II)$

$P_{cum}(k) = 1$ No

Yes ←   K ≤ L   No → E X I T

Yes
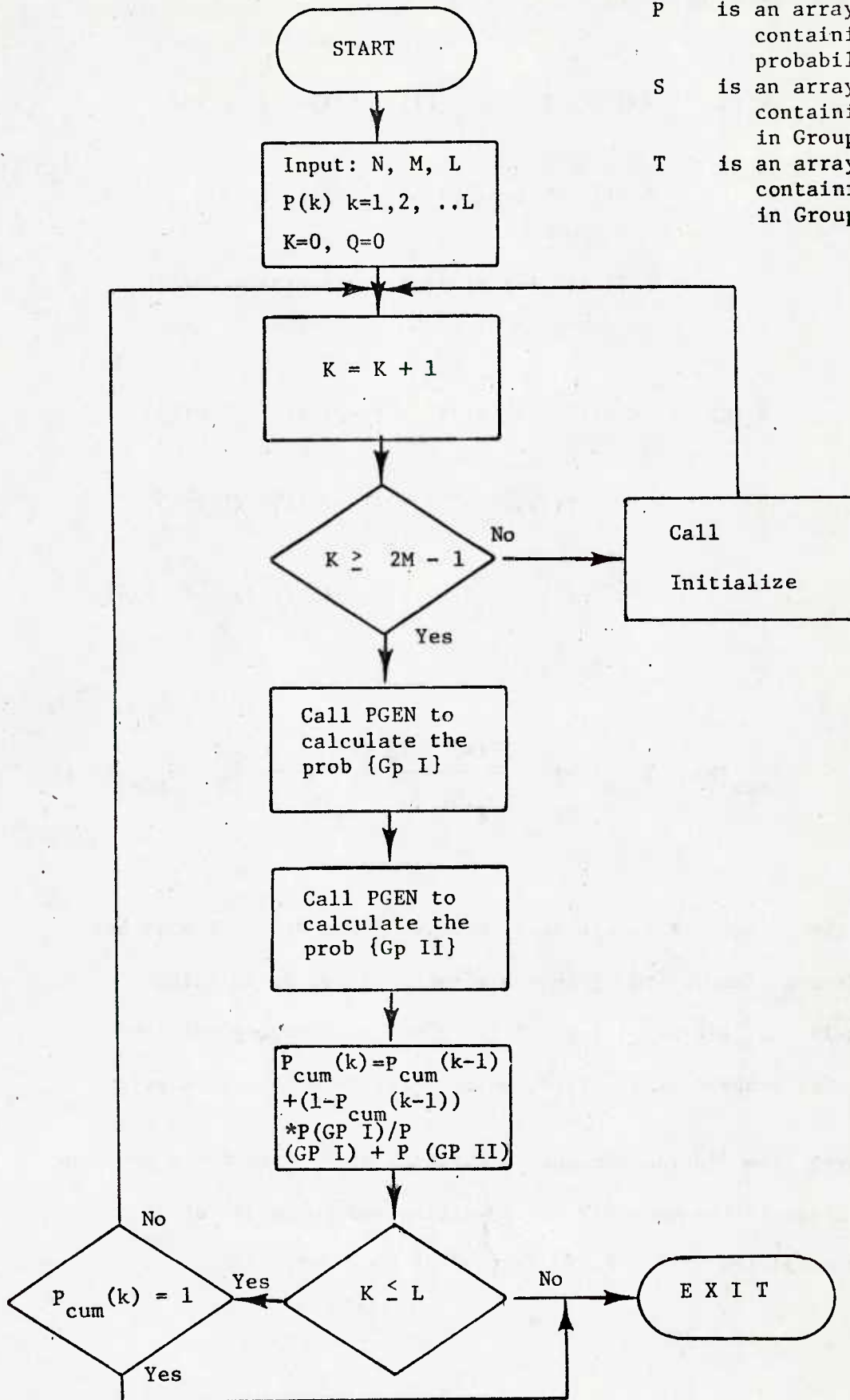
SUBROUTINE INITIALIZE

Initialize generates the
sequences in Group I and
Group II and stores them
in the S & T a rrays
respectively. For 1
$1 \le k \le 2m-1$, $P_{cum}(k)$
is calculated.

```
              ENTRY

    Input: K, N, M, Q
              U
    P(k), k=1, 2, L
```

```
    k = 1  ──Yes──►  Set  S (1) = 1
                          S (2) = 0
```

```
    Sets S (Q+u) = Tu        Set Q = 2
    for u=1, 2, ..U             U = 0
    Set Q = Q+U
```

```
    U = 0                    N = 1  ──Yes──►
    q = 0
```

```
    q = q + 1                  No

                          P_cum(1) = 0
```

```
    S(q) = S(q) * 2
    S(Q+q) = S(q)+1         P_cum(1) = 0
```

```
    q = Q  ──No
       │
      Yes
       │
       A
```

```
           RETURN
```

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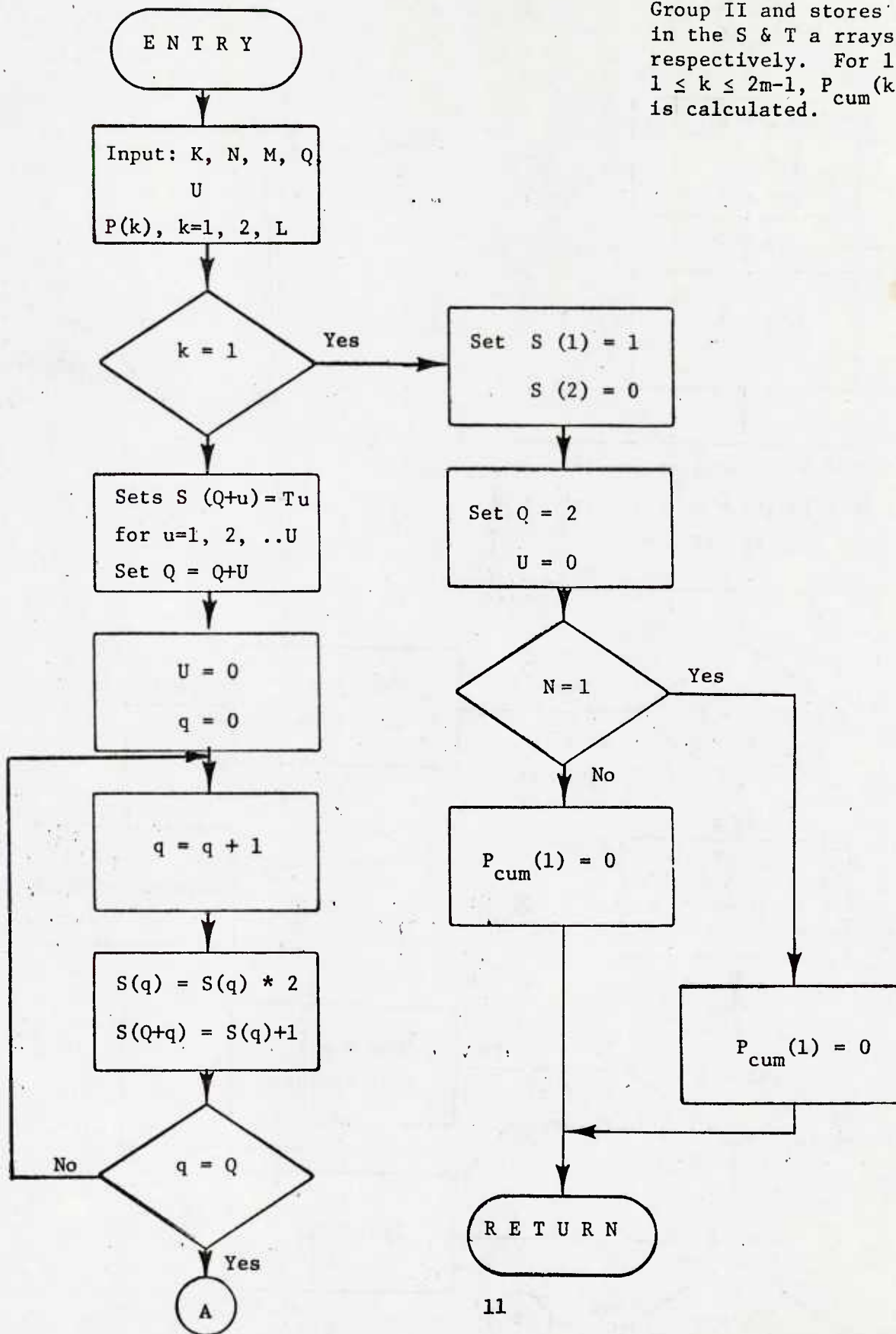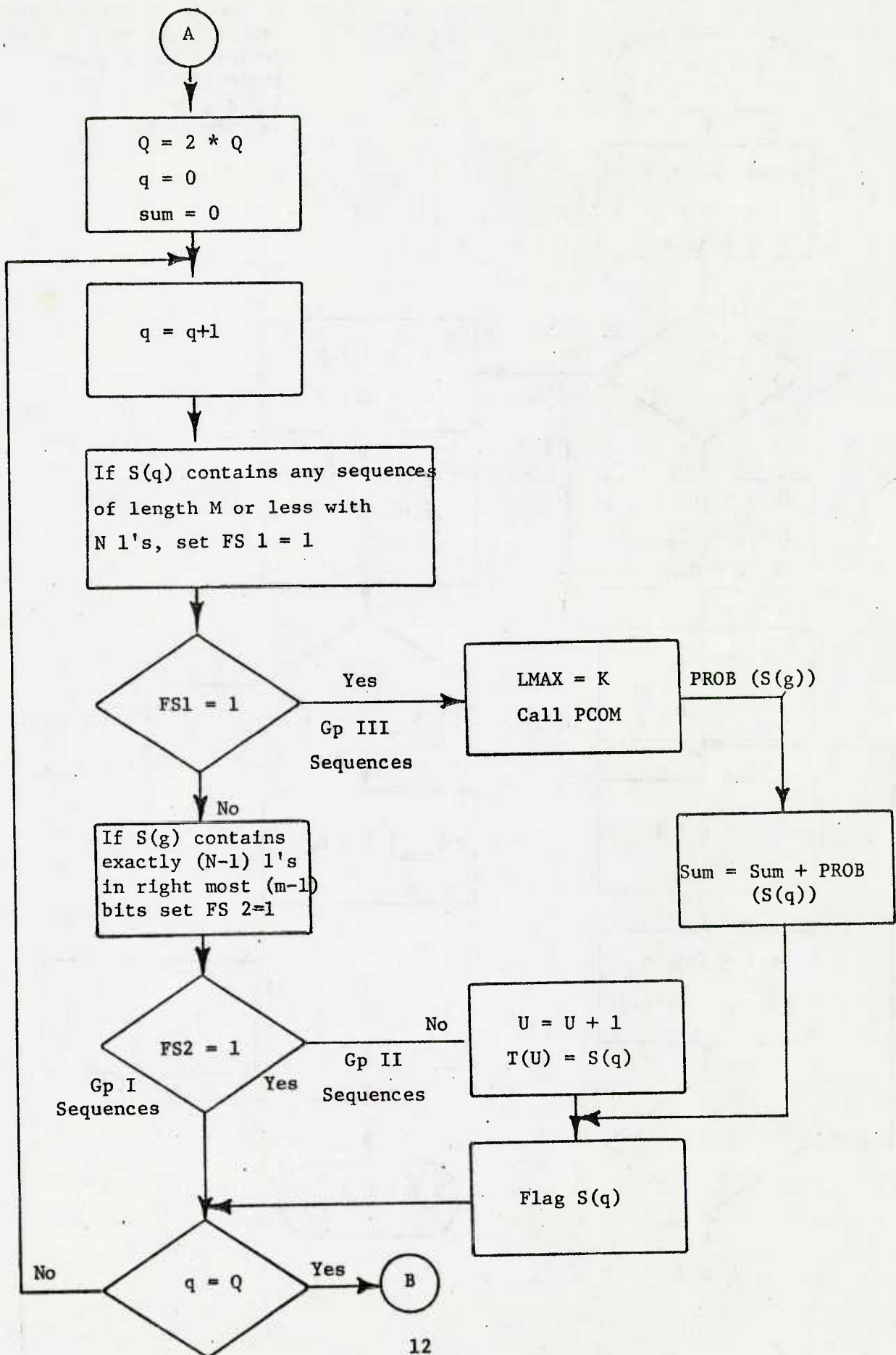
PCOM accepts a binary number of length LMAX and calculates its probability using P(r).

R   is an array of length 2m - 2.

ENTRY

Input: LMAX,
        S, q,
        P(k)
$\ell = 0$

$\ell = \ell + 1$

Test the ℓth bit of S(q)

$S(q, \ell) = 1$

Yes → $R(\ell) = P(k-\ell)$

No

$R(\ell) = 1 - P(k-\ell)$

$\ell = LMAX$

No

Yes

PROB $(S(g)) = $ LMAX
$\Pi R(\ell)$
$\ell = 1$

RETURN

```
          LMAX
              ℓth   1st
          ↓    ↓     ↓
S (q) = 0110101100
```
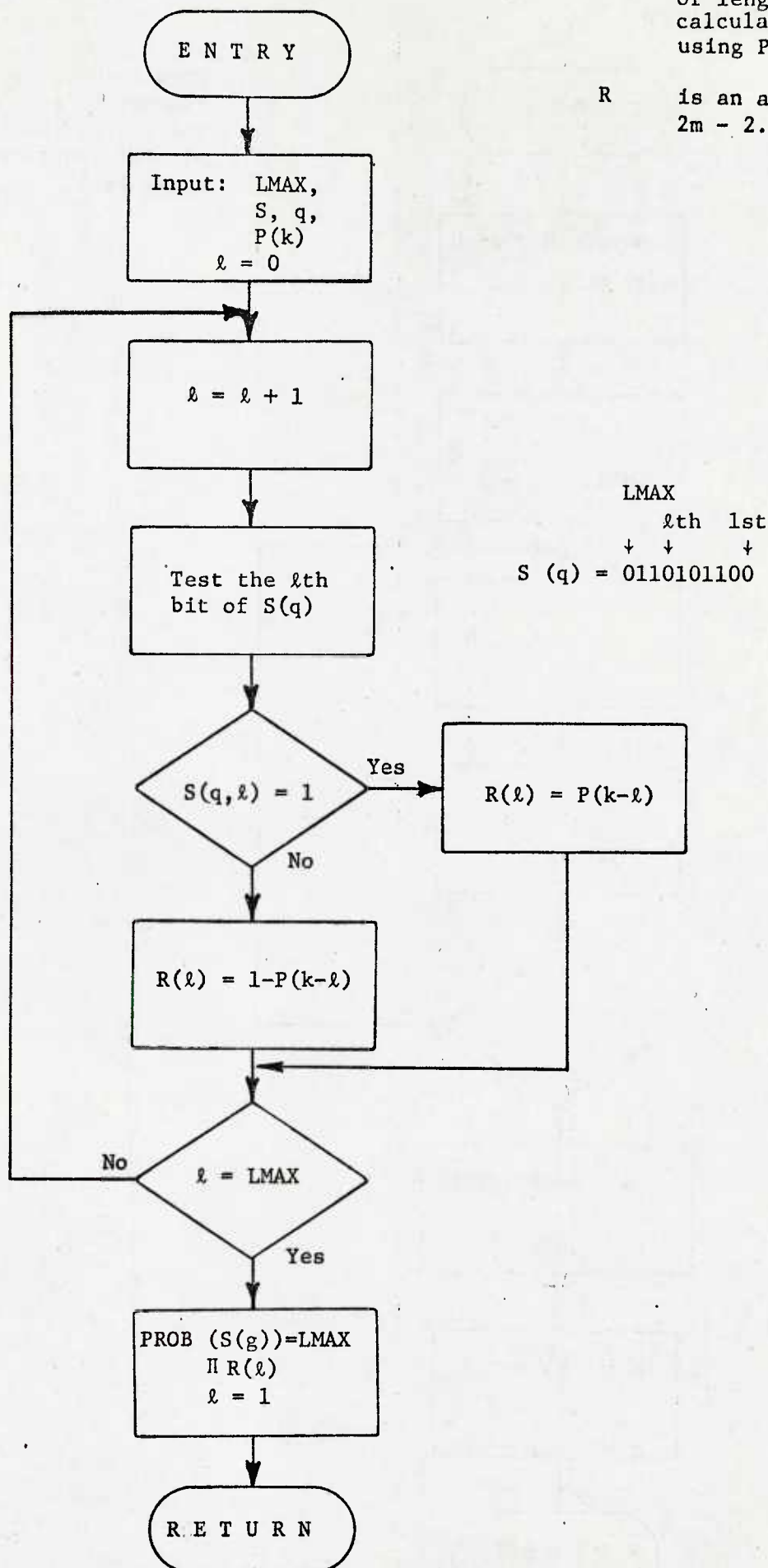
SUBROUTINE PGEN

PGEN accepts an array sequence (either S or T) and calculates the sum of the probability of all the sequences in the array using P(k).

```
        ┌──────────────┐
        │   E N T R Y   │
        └──────────────┘
               │
               ▼
     ┌──────────────────┐
     │ Input: N,M,Q or U │
     │ S(q) or T(n),     │
     │ P(k)              │
     └──────────────────┘
               │
               ▼
        ┌──────────────┐
        │   q = 0       │
        │ LMAX = 2m - 2 │
        └──────────────┘
               │
               ▼◄──────────────────┐
        ┌──────────────┐           │
        │               │           │
        │   q = q+1     │           │
        │               │           │
        └──────────────┘           │
               │                    │
               ▼                    │
        ┌──────────────┐           │
        │               │           │
        │  Call PCOM    │           │
        │               │           │
        └──────────────┘           │
               │                    │
               ▼                    │
            ╱──────╲        No      │
           ╱  q = Q  ╲──────────────┘
           ╲        ╱
            ╲──────╱
               │ Yes
               ▼
     ┌──────────────────┐
     │        Q          │
     │ Sum = Σ PROB (S(q))│
     │       q=1         │
     └──────────────────┘
               │
               ▼
     ┌──────────────────┐
     │  P (GP I) = SUM   │
     │       or          │
     │  P (GP II) = SUM  │
     └──────────────────┘
               │
               ▼
        ┌──────────────┐
        │  R E T U R N  │
        └──────────────┘
```

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```
        ┌───┐
        │ B │
        └─┬─┘
          │
          ▼
┌─────────────────────┐
│                     │
│ P    (k) =          │
│  cum                │
│                     │
│ P    (k-1) + SUM    │
│  cum                │
│                     │
└──────────┬──────────┘
           │
           ▼
┌─────────────────────┐
│ Drop all flagged    │
│ sequences in S      │
│ and compact         │
│                     │
└──────────┬──────────┘
           │
           ▼
     ╭───────────────╮
     │  R E T U R N   │
     ╰───────────────╯
```

$P_{cum}(k) = P_{cum}(k-1) + SUM$

Drop all flagged sequences in S and compact

RETURN